A Bipolar Framework for Combining Beliefs about Vague Concepts

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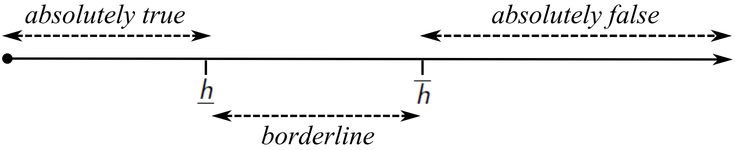
Outline of the Talk

Vagueness in Consensus Modelling

- In many decision making and negotiation scenarios intelligent agents need to arrive at a common shared position or viewpoint concerning a set of relevant propositions.
- One route to consensus is for each agent to adopt a more vague interpretation of underlying concepts so as to soften directly conflicting opinions.
- A defining feature of vague concepts is their admittance of borderline cases which neither absolutely satisfy the concept nor its negation.
- Borderlines can be exploited to provide additional flexibility when combining different, and possibly inconsistent, viewpoints and beliefs in order to achieve consensus.
- Two inconsistent points of view each giving different truth values to a certain proposition p might be combined into a compromise position in which p is a borderline case.

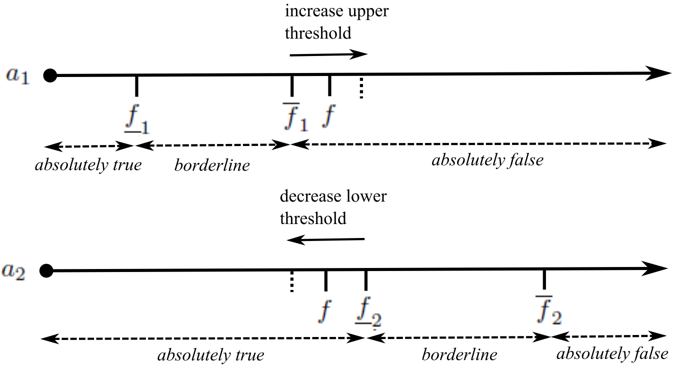
Truth-gaps and Indeterminism

- Propositions can be *absolutely true* or *absolutely false* but there may be also be a *truth-gap*.
- Some propositions may be neither absolutely true nor absolutely false i.e. indeterminate or borderline
- For example, consider the proposition $p =$'Ethel is short'.
- In this context *short* could be defined by two height thresholds $\underline{h} \leq \overline{h}$.
- Let Ethel's height be h. Then p is absolutely true if $h \leq \underline{h}$, absolutely false of $h > \overline{h}$ and borderline otherwise.



Indeterminism and Consensus

- Consider a scenario in which two agents a_1 and a_2 need to agree about the proposition $p =$'*UK inflation is currently low*'.
- Let the actual level of UK inflation be denoted by f.
- Each agent a_i defines lower and upper thresholds $\underline{f}_i \leq \overline{f}_i$.
- p is *absolutely true* if $f \leq \underline{f}_i$, *absolutely false* if $f > \overline{f}_i$ and *borderline* if $\underline{f}_i < f \leq \overline{f}_i$.



Kleene Valuation Pairs

- $\mathcal{L}$ is a language with propositional variable $\mathcal{P} = \{p_1, \ldots, p_l\}$, connectives $\neg, \wedge, \vee$ and sentences $S\mathcal{L}$.

Definition

A valuation pair is a pair of binary functions $(\underline{v}, \overline{v})$ *where* $\underline{v} : S\mathcal{L} \to \{0,1\}$, $\overline{v} : S\mathcal{L} \to \{0,1\}$ *and* $\underline{v} \leq \overline{v}$.

- For sentence $\theta \in S\mathcal{L}$, $\underline{v}(\theta) = 1$ means that θ is *absolutely true*, and $\overline{v}(\theta) = 1$ means that θ is *not absolutely false*.

Definition

A Kleene valuation pair is valuation pair $(\underline{v}, \overline{v})$ *satisfying:*
$\forall \theta, \varphi \in S\mathcal{L}$

1. $\underline{v}(\neg\theta) = 1 - \overline{v}(\theta)$, $\overline{v}(\neg\theta) = 1 - \underline{v}(\theta)$
2. $\underline{v}(\theta \wedge \varphi) = \min(\underline{v}(\theta), \underline{v}(\varphi))$ *and* $\overline{v}(\theta \wedge \varphi) = \min(\overline{v}(\theta), \overline{v}(\varphi))$
3. $\underline{v}(\theta \vee \varphi) = \max(\underline{v}(\theta), \underline{v}(\varphi))$ *and* $\overline{v}(\theta \vee \varphi) = \max(\overline{v}(\theta), \overline{v}(\varphi))$

A Convenient Characterisation

- Kleene valuation pairs can be characterised by a pair of sets of propositional variables (P, N) where $P = \{p_i : \underline{v}(p_i) = 1\}$ and $N = \{p_i : \underline{v}(\neg p_i) = 1\}$
- Notice that since $\underline{v} \leq \overline{v}$ then $\underline{v}(p_i) = 1 \Rightarrow \overline{v}(p_i) = 1 \Rightarrow \underline{v}(\neg p_i) = 0$.
- Hence, $P \cap N = \emptyset$ i.e (P, N) is an *orthopair*.
- Example: $\underline{v}(p_i) = 1$ iff $p_i \in P$, $\overline{v}(p_i) = 1$ iff $p_i \notin N$, $\underline{v}(\neg p_i) = 1$ iff $p_i \in N$, $\overline{v}(\neg p_i) = 1$ iff $p_i \notin P$, $\underline{v}(p_i \wedge \neg p_j) = 1$ iff $p_i \in P$ and $p_j \in N$ and $\overline{v}(p_i \wedge \neg p_j) = 1$ iff $p_i \notin N$ and $p_j \notin P$.
- If $p_i \in (P \cup N)^c$ then $\vec{v}(p_i) = (0, 1)$ i.e. p_i is a borderline proposition.
- Hence, $|(P \cup N)^c|$ quantifies the vagueness of a particular valuation pair.

Consistency and Truth-gaps

- In classical (two-valued) logic two different valuations are by definition inconsistent.
- The presence of truth-gaps allows for valuation pairs which differ but which are consistent in the following weaker sense.
- *Two valuations are consistent providing that, if a sentence is absolutely true according to one valuation it is not absolutely false according to the other, and vice versa.*

Definition

Kleene valuation pairs $\vec{v}_1$ *and* $\vec{v}_2$ *are consistent if and only if* $\forall\theta \in S\mathcal{L}$,

$$\min(\max(\overline{v_1}(\neg\theta), \overline{v_2}(\theta)), \max(\overline{v_2}(\neg\theta), \overline{v_1}(\theta))) = 1$$

Theorem

Valuation pairs $\vec{v}_1$ *and* $\vec{v}_2$ *are consistent if and only if* $P_1 \cap N_2 = P_2 \cap N_1 = \emptyset$.

Semantic Precision: An Ordering on Vagueness

- Let $\mathbb{V}$ be the set of Kleene valuation pairs on $\mathcal{L}$
- We define a semantic precision ordering on $\mathbb{V}$ whereby if $\vec{v}_1 \preceq \vec{v}_2$ then $\vec{v}_1$ tends to classify more sentences of $\mathcal{L}$ as *borderline* than $\vec{v}_2$.

Definition

$\vec{v}_1 \preceq \vec{v}_2$ *iff* $\forall \theta \in S\mathcal{L}$, $\underline{v_1}(\theta) \leq \underline{v_2}(\theta)$ *and* $\overline{v_1}(\theta) \geq \overline{v_2}(\theta)$.

- If $\vec{v}_1 \preceq \vec{v}_2$ then $\vec{v}_1$ tends to classify more sentences of $\mathcal{L}$ as *borderline* than $\vec{v}_2$.
- In other words, one might think of $\preceq$ as ordering valuation pairs according to their relative vagueness.

Theorem

$\vec{v}_1 \preceq \vec{v}_2$ *iff* $P_1 \subseteq P_2$ *and* $N_1 \subseteq N_2$.

- Notice that if $\vec{v}_1 \preceq \vec{v}_2$ then $\vec{v}_1$ and $\vec{v}_2$ are consistent.

Combining Beliefs

- We use the orthopair representation to define four operators for combining valuation pairs.

Definition

- *Given valuation pairs* $\vec{v}_1$ *and* $\vec{v}_2$ *with orthopairs* (P_1, N_1) *and* (P_2, N_2) *respectively, then:*
- **Conservative Combination:** $\vec{v}_1 \otimes \vec{v}_2$ *is defined by the orthopair* $(P_1 \cap P_2, N_1 \cap N_2)$.
- **Optimistic Combination:** *For* $\vec{v}_1$ *and* $\vec{v}_2$ *consistent* $\vec{v}_1 \oplus \vec{v}_2$ *is defined by the orthopair* $(P_1 \cup P_2, N_1 \cup N_2)$.
- **Difference:** $\vec{v}_1 \ominus \vec{v}_2$ *is defined by the orthopair* $(P_1 - N_2, N_1 - P_2)$.
- **Consensus Combination:** $\vec{v}_1 \odot \vec{v}_2$ *is defined by the orthopair* $((P_1 - N_2) \cup (P_2 - N_1), (N_1 - P_2) \cup (N_2 - P_1))$.

Conservative Combination Operator

- $\vec{v}_1 \otimes \vec{v}_2$ is the *most semantically precise* valuation pair which is *less precise* than both $\vec{v}_1$ and $\vec{v}_2$.
- Hence, for all sentences θ, $\underline{v_1 \otimes v_2}(\theta) \leq \min(\underline{v}_1(\theta), \underline{v}_2(\theta))$ and $\overline{v_1 \otimes v_2}(\theta) \geq \max(\overline{v}_1(\theta), \overline{v}_2(\theta))$
- For *literals* the above inequalities can be replaced with *equalities*.
- $\otimes$ is a conservative operator in the sense that, for any sentence θ for which $\vec{v}_1$ and $\vec{v}_2$ have different values, $\vec{v}_1 \otimes \vec{v}_2$ returns a borderline truth-value for θ.

Theorem

$\forall \theta \in S\mathcal{L}$, *if* $\vec{v}_1(\theta) \neq \vec{v}_2(\theta)$ *then* $\vec{v}_1 \otimes \vec{v}_2(\theta) = (0, 1)$

Optimistic Combination Operator

- Optimistic combination can only be applied if $\vec{v}_1$ and $\vec{v}_2$ are consistent.
- For inconsistent valuation pairs $(P_1 \cup P_2, N_1 \cup N_2)$ is not an orthopair since $(P_1 \cup P_2) \cap (N_1 \cup N_2) \neq \emptyset$.
- For consistent valuation pairs $\vec{v}_1 \oplus \vec{v}_2$ is the *least semantically precise* valuation pair which is *more precise* than both $\vec{v}_1$ and $\vec{v}_2$.
- Hence for any sentence θ, $\underline{v_1 \oplus v_2}(\theta) \geq \max(\underline{v}_1(\theta), \underline{v}_2(\theta))$ and $\overline{v_1 \oplus v_2}(\theta) \leq \min(\overline{v}_1(\theta), \overline{v}_2(\theta))$
- Again, by restricting to literals these inequalities can be replaced by equalities.

Difference Operator

- This asymmetric operator is motivated by the idea that in some circumstances an agent may need to minimally adapt their beliefs so that they become consistent with another agent's viewpoint.
- $\vec{v}_1 \ominus \vec{v}_2$ is the *most semantically precise* valuation pair consistent with $\vec{v}_2$, but which is *less semantically precise* than $\vec{v}_1$.

Theorem

$\forall \theta \in S\mathcal{L}$, $\underline{v_1 \ominus v_2}(\theta) \leq \min(\underline{v_1}(\theta), \overline{v_2}(\theta))$ *and* $\overline{v_1 \ominus v_2}(\theta) \geq \max(\overline{v}_1(\theta), \underline{v_2}(\theta))$.

- By restricting to literals these inequalities can be replaced by equalities

Consensus Operator

- This operator combines two valuations to obtain a new consensus viewpoint even when the original valuations are inconsistent, and which is more semantically precise than that obtained from the conservative operator.
- The consensus operator consists of a form of *contraction* whereby the two agents *minimally soften* their viewpoints until they become consistent, followed by a form of *expansion* where the two viewpoints are added together using the optimistic combination operator.

Theorem

$\vec{v}_1 \odot \vec{v}_2 = (\vec{v}_1 \ominus \vec{v}_2) \oplus (\vec{v}_2 \ominus \vec{v}_1)$

The Operators Applied to Literals

- Let $\mathbf{t} = (1, 1)$ as absolutely true, $\mathbf{b} = (0, 1)$ as borderline and $\mathbf{f} = (0, 0)$ as absolutely false.
- We can then express the four operators as truth-tables:

$\otimes$	**t**	**b**	**f**
t	**t**	**b**	**b**
b	**b**	**b**	**b**
f	**b**	**b**	**f**

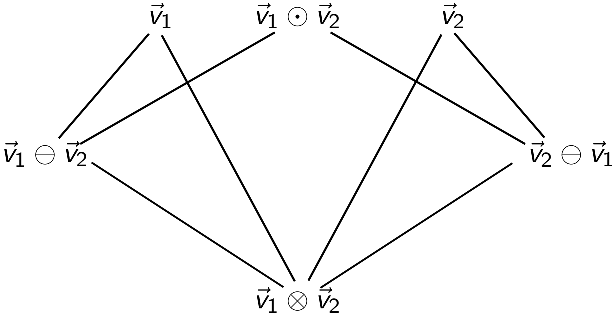
$\oplus$	**t**	**b**	**f**
t	**t**	**t**	—
b	**t**	**b**	**f**
f	—	**f**	**f**

$\ominus$	**t**	**b**	**f**
t	**t**	**b**	**b**
b	**t**	**b**	**f**
f	**b**	**b**	**f**

$\odot$	**t**	**b**	**f**
t	**t**	**t**	**b**
b	**t**	**b**	**f**
f	**b**	**f**	**f**

Hasse Diagram of Operators

- Below is a Hasse diagram showing the relative semantic precision of different combined valuations in comparison with the original valuation pairs $\vec{v}_1$ and $\vec{v}_2$.
- $\vec{v}_1 \oplus \vec{v}_2$ is not shown on the diagram since if $\vec{v}_1$ and $\vec{v}_2$ are consistent then $\vec{v}_1 \ominus \vec{v}_2 = \vec{v}_1$, $\vec{v}_2 \ominus \vec{v}_1 = \vec{v}_2$ and $\vec{v}_1 \odot \vec{v}_2 = \vec{v}_1 \oplus \vec{v}_2$.



Uncertainty and Kleene Belief Pairs

- Within the proposed bipolar framework, uncertainty concerning the sentences of $\mathcal{L}$ effectively corresponds to uncertainty as to which is the correct Kleene valuation pair for $\mathcal{L}$.
- In the following we assume that this uncertainty is quantified by a probability measure w on the set of Kleene valuation pairs $\mathbb{V}$.

Definition

Let $\mathbb{V}$ be the set of all Kleene valuation pairs on $\mathcal{L}$ and let w be a probability distribution defined on $\mathbb{V}$ so that $w(\vec{v})$ is the agent's subjective belief that $\vec{v}$ is the true valuation pair for $\mathcal{L}$. Then $\vec{\mu}_w = (\underline{\mu}_w, \overline{\mu}_w)$ is a Kleene belief pair where $\forall \theta \in S\mathcal{L}$,
$\underline{\mu}_w(\theta) = w(\{\vec{v} \in \mathbb{V} : \underline{v}(\theta) = 1\})$ *and*
$\overline{\mu}_w(\theta) = w(\{\vec{v} \in \mathbb{V} : \overline{v}(\theta) = 1\})$.

Combination of Belief Pairs

- Suppose we have two agents with beliefs about $S\mathcal{L}$ quantified by Kleene belief pairs $\vec{\mu}_{w_1}$ and $\vec{\mu}_{w_2}$ respectively.
- The following definition proposes how the conservative operator can be extended to this case, as well as providing an exemplar of a general scheme which can then be employed to extend other combination operators for valuation pairs to belief pairs.

Definition

A conservative combination of Kleene belief pairs $\vec{\mu}_{w_1}$ and $\vec{\mu}_{w_2}$ is a belief pair $\vec{\mu}_{w_1 \otimes_q w_2}$ where $w_1 \otimes_q w_2$ is a probability distribution on $\mathbb{V}$ for which

$$w_1 \otimes_q w_2(\vec{v}) = q(\{(\vec{v_1}, \vec{v_2}) : \vec{v}_1 \otimes \vec{v}_2 = \vec{v}\})$$

where q is any 2-dimensional probability distribution on $\mathbb{V} \times \mathbb{V}$ with marginals w_1 and w_2.

General Scheme for Combination Operators on Belief Pairs

- The above definition provides a general scheme for extending combination operators from valuation pairs to belief pairs.
- The joint distribution q should be viewed as an integral part of belief combination.
- Potentially, q could be agreed as the result of negotiation between the two agents.
- $q(\vec{v}_1, \vec{v}_2)$ is the agreed probability weighting allocated by the two agents specifically to the combination of $\vec{v}_1$ and $\vec{v}_2$.
- The *independent interaction model* $q = w_1 \times w_2$ corresponds to the case of minimal interaction between the two agents.
- We might think of each agent as independently identifying valuation pairs to combine.

Bounds for Three Operators under Uncertainty

Theorem

- $\underline{\mu}_{w_1 \otimes_q w_2}(\theta) \leq \min(\underline{\mu}_{w_1}(\theta), \underline{\mu}_{w_2}(\theta))$ *and*
  $\overline{\mu}_{w_1 \otimes_q w_2}(\theta) \geq \max(\overline{\mu}_{w_1}(\theta), \overline{\mu}_{w_2}(\theta))$
- $\underline{\mu}_{w_1 \ominus_q w_2}(\theta) \leq \min(\underline{\mu}_{w_1}(\theta), \overline{\mu}_{w_2}(\theta))$ *and*
  $\overline{\mu}_{w_1 \ominus_q w_2}(\theta) \geq \max(\overline{\mu}_{w_1}(\theta), \underline{\mu}_{w_2}(\theta))$
- $\underline{\mu}_{w_1 \odot_q w_2}(\theta) \geq \max(\underline{\mu}_{w_1 \ominus_q w_2}(\theta), \underline{\mu}_{w_2 \ominus_q w_1}(\theta))$ *and*
  $\overline{\mu}_{w_1 \odot_q w_2}(\theta) \leq \min(\overline{\mu}_{w_1 \ominus_q w_2}(\theta), \overline{\mu}_{w_2 \ominus_q w_1}(\theta))$

Theorem

If $q = w_1 \times w_2$ *then*

- $\underline{\mu}_{w_1 \otimes_q w_2}(\theta) \leq \underline{\mu}_{w_1}(\theta) \times \underline{\mu}_{w_2}(\theta)$ *and*
  $\overline{\mu}_{w_1 \otimes_q w_2}(\theta) \geq \overline{\mu}_{w_1}(\theta) + \overline{\mu}_{w_2}(\theta) - \overline{\mu}_{w_1}(\theta) \times \overline{\mu}_{w_2}(\theta)$
- $\underline{\mu}_{w_1 \ominus_q w_2}(\theta) \leq \underline{\mu}_{w_1}(\theta) \times \overline{\mu}_{w_2}(\theta)$ *and*
  $\overline{\mu}_{w_1 \ominus_q w_2}(\theta) \geq \overline{\mu}_{w_1}(\theta) + \underline{\mu}_{w_2}(\theta) - \overline{\mu}_{w_1}(\theta) \times \underline{\mu}_{w_2}(\theta)$

Results for Literals

- For the *independent interaction model*, if we restrict ourselves to *literals* we obtain the following equalities:

Theorem

If $q = w_1 \times w_2$ *then*

- $\underline{\mu}_{w_1 \otimes_q w_2}(l) = \underline{\mu}_{w_1}(l) \times \underline{\mu}_{w_2}(l)$ *and*
  $\overline{\mu}_{w_1 \otimes_q w_2}(l) = \overline{\mu}_{w_1}(l) + \overline{\mu}_{w_2}(l) - \overline{\mu}_{w_1}(l) \times \overline{\mu}_{w_2}(l)$
- $\underline{\mu}_{w_1 \ominus_q w_2}(l) = \underline{\mu}_{w_1}(l) \times \overline{\mu}_{w_2}(l)$ *and*
  $\overline{\mu}_{w_1 \ominus_q w_2}(l) = \overline{\mu}_{w_1}(l) + \underline{\mu}_{w_2}(l) - \overline{\mu}_{w_1}(l) \times \underline{\mu}_{w_2}(l)$
- $\underline{\mu}_{w_1 \odot_q w_2}(l) = \underline{\mu}_{w_1}(l) \times \overline{\mu}_{w_2}(l) + \overline{\mu}_{w_1}(l) \times \underline{\mu}_{w_2}(l) - \underline{\mu}_{w_1}(l) \times \underline{\mu}_{w_2}(l)$
  and
  $\overline{\mu}_{w_1 \odot_q w_2}(l) = \underline{\mu}_{w_1}(l) + \underline{\mu}_{w_2}(l) + \overline{\mu}_{w_1}(l) \times \overline{\mu}_{w_2}(l) - \overline{\mu}_{w_1}(l) \times \underline{\mu}_{w_2}(l) - \underline{\mu}_{w_1}(l) \times \overline{\mu}_{w_2}(l)$

Comments and Conclusions

- We have outlined a bipolar framework for combining potentially inconsistent beliefs which exploits the inherent vagueness of concepts in natural language.
- Kleene belief pairs have been introduced as quantitative lower and upper measures of belief which incorporate both semantic uncertainty and indeterminism.
- We have then proposed a general scheme for extending combination operators on valuation pairs to belief pairs.
- *Comments:* This scheme can also be applied to the optimistic operator but with the addition of a normalizing factor to take account of inconsistencies between valuation pairs.
- While here we are using valuation pairs to model vagueness, the literature contains many proposals to use three-valued logic to represent partial knowledge.
- In forthcoming work we will be looking at the proposed combination operators in this context and comparing it with Boolean possibility theory.