Borderlines and Probabilities of Borderlines

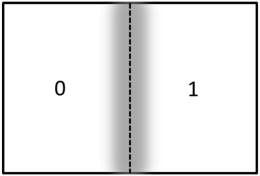
Jonathan Lawry

Department of Engineering Mathematics, University of Bristol, UK

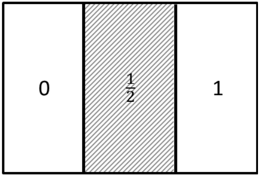
Progic 2015
University of Kent, 22 April - 24 April 2015

What is Vagueness?

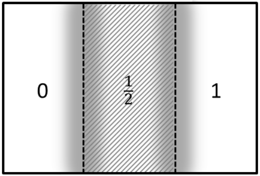
Today, vague predicates are standardly characterized by three main 'symptoms', namely as predicates that are sorities susceptible, that have borderline cases, and that have blurry boundaries. (Paul Égré)



Blurred boundaries

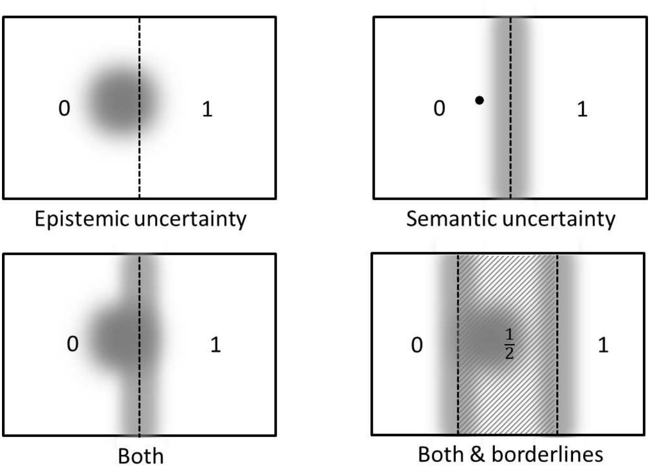


Borderline cases



An integrated model

Probability and Vagueness



Borderline Cases

$\mathcal{L}$ is a language of propositional logic with propositional variables $\mathcal{P} = \{p_1, \ldots, p_n\}$, connectives $\neg, \wedge, \vee$ and sentences $S\mathcal{L}$.

Definition

Three Valued Valuation:
A three valued valuation on $\mathcal{L}$ is a function $\mathbf{v} : S\mathcal{L} \to \{1, \frac{1}{2}, 0\}$ such that $\forall \theta, \varphi \in S\mathcal{L}$ if $\mathbf{v}(\theta) \in \{0, 1\}$ and $\mathbf{v}(\varphi) \in \{0, 1\}$ then
$\mathbf{v}(\neg\theta) = 1 - \mathbf{v}(\theta)$,
$\mathbf{v}(\theta \wedge \varphi) = \min(\mathbf{v}(\theta), \mathbf{v}(\varphi))$ and
$\mathbf{v}(\theta \vee \varphi) = \max(\mathbf{v}(\theta), \mathbf{v}(\varphi))$.
Here the truth values denote absolutely true (1), borderline ($\frac{1}{2}$) and absolutely false (0) respectively. The restriction on $\mathbf{v}$ is that it should obey the same rules as Tarski valuations in the case of Boolean expressions.

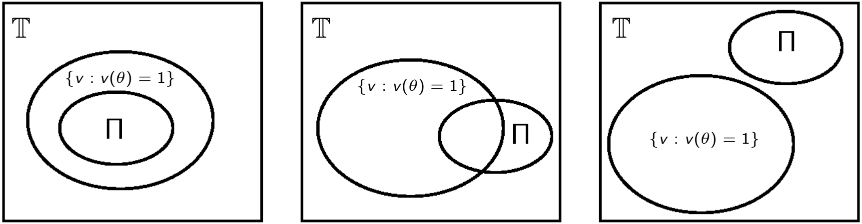
Supervaluations

Definition

Let $\mathbb{T}$ denote the set of Tarski valuations defined on $\mathcal{L}$. A supervaluation is a three valued valuation define by a set $\Pi \subseteq \mathbb{T}$ corresponding to admissible precisifications, such that $\forall \theta \in S\mathcal{L}$;

$$\mathbf{v}(\theta) = \begin{cases} 1 : \min\{v(\theta) : v \in \Pi\} = 1 \\ 0 : \max\{v(\theta) : v \in \Pi\} = 0 \\ \frac{1}{2} : \text{otherwise} \end{cases}$$

Let $\mathbb{S}$ be the set of supervaluations on $\mathcal{L}$.



$\mathbf{v}(\theta) = 1$ $\mathbf{v}(\theta) = \frac{1}{2}$ $\mathbf{v}(\theta) = 0$

Kleene Valuations

Definition

A Kleene valuation is a three valued valuation defined recursively such that $\forall\theta, \varphi \in S\mathcal{L}$;
$\mathbf{v}(\neg\theta) = 1 - \mathbf{v}(\theta)$,
$\mathbf{v}(\theta \wedge \varphi) = \min(\mathbf{v}(\theta), \mathbf{v}(\varphi))$ and
$\mathbf{v}(\theta \vee \varphi) = \max(\mathbf{v}(\theta), \mathbf{v}(\varphi))$

Let $\mathbb{K}$ be the set of Kleene valuations on $\mathcal{L}$.

$\neg$	1	0
	$\frac{1}{2}$	$\frac{1}{2}$
	0	1

$\wedge$	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	0
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0
0	0	0	0

$\vee$	1	$\frac{1}{2}$	0
1	1	1	1
$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$
0	1	$\frac{1}{2}$	0

Table: Kleene truth tables

An Integrated Approach

- In the context of a three valued truth model we quantify uncertainty in terms of a probability distribution w over a finite set of possible three valued valuations $\mathbb{V}$.
- This naturally generates the following three measures on $S\mathcal{L}$:
- *Lower measures*

$$\underline{\mu}(\theta) = w(\{\mathbf{v} \in \mathbb{V} : \mathbf{v}(\theta) = 1\})$$

- *Upper measures*

$$\overline{\mu}(\theta) = w(\{\mathbf{v} \in \mathbb{V} : \mathbf{v}(\theta) \neq 0\})$$

- *Truth degrees* (bad name)

$$td(\theta) = \frac{\underline{\mu}(\theta) + \overline{\mu}(\theta)}{2}$$

$$= w(\{\mathbf{v} \in \mathbb{V} : \mathbf{v}(\theta) = 1\}) + \frac{w(\{\mathbf{v} \in \mathbb{V} : \mathbf{v}(\theta) = \frac{1}{2}\})}{2}$$

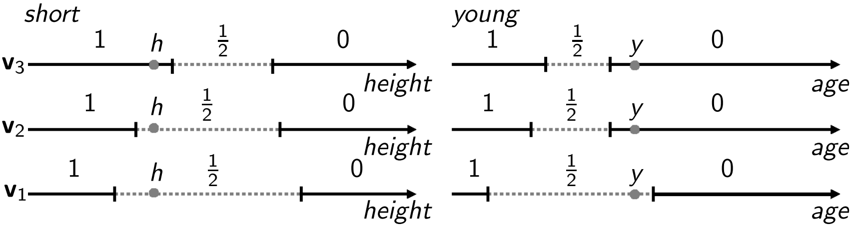
Supervaluation and Kleene Belief Pairs

- If *w* is defined on $\mathbb{V} \subseteq \mathbb{S}$ or $\mathbb{V} \subseteq \mathbb{K}$ then we refer to $\vec{\mu} = (\underline{\mu}, \overline{\mu})$ as a *supervaluation belief pair* or a *Kleene belief pair*.
- Both satisfy duality: $\underline{\mu}(\neg\theta) = 1 - \overline{\mu}(\theta)$.
- It is well known (Jaffray, Field) that for supervaluation belief pairs $\underline{\mu}$ and $\overline{\mu}$ are, respectively, Dempster-Shafer belief and plausibility measures on $S\mathcal{L}$.
- Hence, supervaluation beliefs pairs preserve classical equivalences and tautologies, while $\underline{\mu}$ is super-additive and $\overline{\mu}$ is sub-additive.
- In contrast both lower and upper Kleene measures (Lawry, Williams, ?) are additive, but do not preserve classical tautologies and equivalences.

A Vagueness Ordering

Definition

Semantic Precision:
For three valued valuations $\mathbf{v}_1, \mathbf{v}_2$, $\mathbf{v}_1 \preceq \mathbf{v}_2$ then $\mathbf{v}_1 \preceq \mathbf{v}_2$ if and only if $\forall \theta \in S\mathcal{L}$, $\mathbf{v}_1(\theta) = 1 \Rightarrow \mathbf{v}_2(\theta) = 1$ and $\mathbf{v}_1(\theta) = 0 \Rightarrow \mathbf{v}_2(\theta) = 0$.



$\mathbf{v}_1 \preceq \mathbf{v}_2 \preceq \mathbf{v}_3$.

Probability and Fuzzy Logic

Fuzziness is probability in disguise. I can design a controller with probability that could do the same thing that you could do with fuzzy logic. [Myron Tribus]

- $\mathcal{L}$ is a language with propositional variable $\mathcal{P} = \{p_1, \ldots, p_n\}$, connectives $\neg, \wedge, \vee$ and sentences $S\mathcal{L}$.
- Consider fuzzy truth degrees and probability measures defined on $S\mathcal{L}$:

- *Fuzzy Truth Degree*: $\forall \theta, \varphi \in S\mathcal{L}$;
- $\zeta(\neg\theta) = 1 - \zeta(\theta)$.
- $\zeta(\theta \wedge \varphi) = \min(\zeta(\theta), \zeta(\varphi))$.
- $\zeta(\theta \vee \varphi) = \max(\zeta(\theta), \zeta(\varphi))$.

- *Probability Measure*: $\forall \theta, \varphi \in S\mathcal{L}$;
- If $\models \theta$ then $\mu(\theta) = 1$.
- If $\theta \equiv \varphi$ then $\mu(\theta) = \mu(\varphi)$.
- If $\models \neg(\theta \wedge \varphi)$ then $\mu(\theta \vee \varphi) = \mu(\theta) + \mu(\varphi)$

Probability in Disguise

Theorem

Let w be a probability distribution on $\mathbb{K}$ such that $\{v : w(v) > 0\} = \{v_1, \ldots, v_m\}$ where $v_1 \preceq \ldots \preceq v_m$ and let $\underline{\mu}$ and $\overline{\mu}$ be the associated lower and upper beliefs on $S\mathcal{L}$. Also, let $td : S\mathcal{L} \to [0, 1]$ be the truth degree defined as the average of $\underline{\mu}$ and $\overline{\mu}$. Then in this case td is a fuzzy truth degree on $S\mathcal{L}$.

Theorem

For any fuzzy truth degree ζ on $S\mathcal{L}$, there is a unique sequence $v_1 \preceq \ldots \preceq v_m$ of Kleene valuations on $\mathcal{L}$ and an associated probability distribution w on $\mathbb{K}$ for which $\{v : w(v) > 0\} = \{v_1, \ldots, v_m\}$, such that $\forall \theta \in S\mathcal{L}$;

$$\zeta(\theta) = td(\theta) = \frac{\underline{\mu}(\theta) + \overline{\mu}(\theta)}{2}$$

Characterising Borderlines (Supervaluations)

Definition

- **P1 Duality:** $\mathbf{v}(\neg\theta) = 1 - \mathbf{v}(\theta)$.
- **P2 Tautology:** If $\models \theta$ then $\mathbf{v}(\theta) = 1$.
- **P3 Equivalence:** If $\theta \equiv \varphi$ then $\mathbf{v}(\theta) = \mathbf{v}(\varphi)$.

P1 simply requires that the negation of a borderline case is also a borderline case. **P2** and **P3** require respectively that classical (Tarski) tautologies and equivalences are preserved by three valued valuations.

Theorem

Let $\mathbf{v}$ *be a three valued valuation of* $\mathcal{L}$*, then* $\mathbf{v}$ *satisfies* **P1**, **P2** *and* **P3** *if and only if* $\mathbf{v}$ *is a supervaluation.*

Characterising Borderlines (Kleene)

Definition

- **P4 Commutativity:** $\mathbf{v}(\theta \wedge \varphi) = \mathbf{v}(\varphi \wedge \theta)$ and $\mathbf{v}(\theta \vee \varphi) = \mathbf{v}(\varphi \vee \theta)$.
- **P5 Bounds:** If $\mathbf{v}(\theta) \neq 1$ or $\mathbf{v}(\varphi) \neq 1$ then $\mathbf{v}(\theta \wedge \varphi) \neq 1$, and if $\mathbf{v}(\theta) \neq 0$ or $\mathbf{v}(\varphi) \neq 0$ then $\mathbf{v}(\theta \vee \varphi) \neq 0$.
- **P6 Monotonicity:** If $\mathbf{v}(\psi) < \mathbf{v}(\varphi)$ then $\mathbf{v}(\theta \wedge \psi) \leq \mathbf{v}(\theta \wedge \varphi)$ and $\mathbf{v}(\theta \vee \psi) \leq \mathbf{v}(\theta \vee \varphi)$.
- **P7 Borderline:** If $\mathbf{v}(\theta) = \mathbf{v}(\varphi) = \frac{1}{2}$ then $\mathbf{v}(\theta \wedge \varphi) = \mathbf{v}(\theta \vee \varphi) = \frac{1}{2}$.

Theorem

Let $\mathbf{v}$ *be a three valued valuation on* $\mathcal{L}$*, then* $\mathbf{v}$ *satisfies* **P1**, **P4**, **P5**, **P6** *and* **P7** *if and only if* $\mathbf{v}$ *is a Kleene valuation.*

Why Not Lukasiewicz?

$\neg$	1	0
	$\frac{1}{2}$	$\frac{1}{2}$
	0	1

$\wedge$	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	0
$\frac{1}{2}$	$\frac{1}{2}$	0	0
0	0	0	0

$\vee$	1	$\frac{1}{2}$	0
1	1	1	1
$\frac{1}{2}$	1	1	$\frac{1}{2}$
0	1	$\frac{1}{2}$	0

Table: Lukasiewicz truth tables

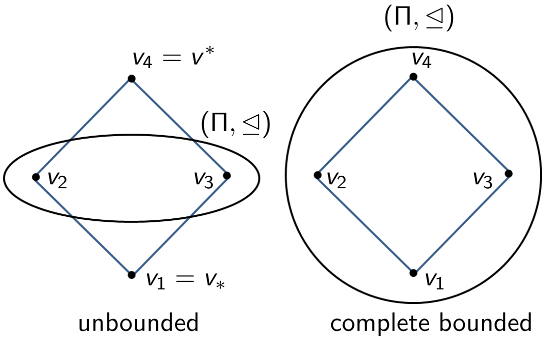
Theorem

Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be Lukasiewicz valuations then $\mathbf{v}_1 \not\prec \mathbf{v}_2$.

Complete Bounded Supervaluations

Definition

Let $\trianglelefteq$ be a partial ordering on $\mathbb{T}$ according to which $v_1 \trianglelefteq v_2$ if and only if $\forall p_i \in \mathcal{P}$, $v_1(p_i) \leq v_2(p_i)$. Then a *complete bounded supervaluation* is a supervaluation with $\Pi = \{v \in \mathbb{T} : v_* \trianglelefteq v \trianglelefteq v^*\}$ where $\forall p_i \in \mathcal{P}$, $v_*(p_i) = \min\{v(p_i) : v \in \Pi\}$ and $v^*(p_i) = \max\{v(p_i) : v \in \Pi\}$.



$v_1(p_1 \wedge \neg p_2 \wedge \neg p_3) = 1$

$v_2(p_1 \wedge p_2 \wedge \neg p_3) = 1$

$v_3(p_1 \wedge \neg p_2 \wedge p_3) = 1$

$v_4(p_1 \wedge p_2 \wedge p_3) = 1$

Relating Kleene and Supervaluations

Definition

$S\mathcal{L}^* \subseteq S\mathcal{L}$ is the subset of the sentences of $\mathcal{L}$ in negated normal form, for which it is not the case that both a propositional variable and its negation appear.

Theorem

Let $\mathbf{v}_{cbs}$ be a complete bounded supervaluation, then there exists a unique Kleene valuation $\mathbf{v}_k$ such that $\mathbf{v}_k \preceq \mathbf{v}_{cbs}$ and $\forall\theta \in S\mathcal{L}^*$, $\mathbf{v}_k(\theta) = \mathbf{v}_{cbs}(\theta)$.

Theorem

Let $\vec{\mu}_1$ be a complete bounded supervaluation belief pair on $S\mathcal{L}$, then there is a Kleene belief pair $\vec{\mu}_2$ on $S\mathcal{L}$ such that $\forall\theta \in S\mathcal{L}^*$, $\vec{\mu}_1(\theta) = \vec{\mu}_2(\theta)$ and $\forall\theta \in S\mathcal{L}$, $\underline{\mu}_1(\theta) \geq \underline{\mu}_2(\theta)$ and $\overline{\mu}_1(\theta) \leq \overline{\mu}_2(\theta)$.

Conclusion

- I have proposed combining three valued valuations and probability to generate lower and upper measures on $S\mathcal{L}$ (belief pairs).
- In a propositional logic setting three valued valuations represent borderline cases, and probability quantifies both epistemic and semantic uncertainty.
- Supervaluation belief pairs are Dempster-Shafer belief and plausibility measures on $S\mathcal{L}$.
- A special case of Kleene belief pairs provide a characterisation of fuzzy truth degrees.
- I have given characterisations of Kleene and supervaluations in terms of axioms defined for a general class of three valued valuations.
- I have shown that there is a close relationship between Kleene and complete bounded supervaluations, and also the associated belief pairs.