A Random Set and Prototype Theory Interpretation of Intuitionistic Fuzzy Sets

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Overview

Intuitionistic Fuzzy Sets

- Intuitionistic fuzzy sets (IFS) were first proposed by Atanassov as a bipolar model of fuzzy sets.
- The basis of IFS are two functions τ and ν quantifying membership and non-membership respectively.
- $\tau_\theta(x)$ corresponds to the membership degree of x in the extension of θ and $\nu_\theta(x)$ is the non-membership degree of x in the extension of θ.
- It is assumed that $\tau_\theta(x) + \nu_\theta(x) \leq 1$ and:
- $\tau_{\theta \wedge \varphi}(x) = \min(\tau_\theta(x), \tau_\varphi(x))$, $\nu_{\theta \wedge \varphi}(x) = \max(\nu_\theta(x), \nu_\varphi(x))$
- $\tau_{\theta \vee \varphi}(x) = \max(\tau_\theta(x), \tau_\varphi(x))$, $\nu_{\theta \vee \varphi}(x) = \min(\nu_\theta(x), \nu_\varphi(x))$

Interval Fuzzy Sets

- There is an isomorphic relationship between IFS and an older notion of interval fuzzy sets (Zadeh, Grattan-Guiness, Jahn, Sambuc).
- Lower and upper membership degrees are defined, where $\underline{\mu}_\theta(x)$ is the lower membership degree of element x in the extension of θ, and $\overline{\mu}_\theta(x)$ is the upper membership degree of x in θ.
- These lower and upper memberships then satisfy the following properties:
- $\underline{\mu}_\theta(x) \leq \overline{\mu}_\theta(x)$
- $\underline{\mu}_{\neg\theta}(x) = 1 - \overline{\mu}_\theta(x)$, and $\overline{\mu}_{\neg\theta}(x) = 1 - \underline{\mu}_\theta(x)$
- $\underline{\mu}_{\theta\wedge\varphi}(x) = \min(\underline{\mu}_\theta(x), \underline{\mu}_\varphi(x))$, $\overline{\mu}_{\theta\wedge\varphi}(x) = \min(\overline{\mu}_\theta(x), \overline{\mu}_\varphi(x))$
- $\underline{\mu}_{\theta\vee\varphi}(x) = \max(\underline{\mu}_\theta(x), \underline{\mu}_\varphi(x))$, $\overline{\mu}_{\theta\vee\varphi}(x) = \max(\overline{\mu}_\theta(x), \overline{\mu}_\varphi(x))$

Prototype Theory

- Prototype theory has been proposed by Rosch as an alternative model of concepts in natural language.
- Concepts are represented by a set of prototypical cases.
- Categorization of elements from the underlying universe Ω is based on similarity to the prototypes as quantified by a distance metric defined on Ω.
- Typicallity is a decreasing function of distance from prototypes, so that some instances are seen as being more typical exemplars of a concept than others.
- This notion of typicality is also strongly related to concept vagueness where borderline cases have an intermediate range of typicality values.

Random Set Theory

- Random set theory has been proposed by Goodman and Nguyen as a framework for linguistic reasoning in rule based systems.
- Stated simply, random sets are set-valued variables with an associated probability measure.
- In Goodman and Nguyen's work they provide a model of vague concepts from the perspective that the extension of such a concept is an uncertain set.
- Given a random set $\mathcal{R}$ modelling a concept, the fuzzy set membership value of an element x in $\mathcal{R}$ is then taken to be the probability that the value of $\mathcal{R}$ is a set which contains x.
- This is the single point coverage function of the random set $\mathcal{R}$.

Preliminaries

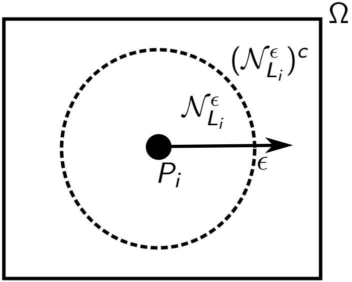
- Let Ω denote the underlying universe of discourse and $LA = \{L_1, \ldots, L_n\}$ be a finite set of labels for describing elements of Ω.
- LE are the set of compound expressions generated by recursive application of the connectives $\wedge$, $\vee$ and $\neg$ to the labels in LA.
- E.g., if LA contains labels *red* and *blue*, then LE contains expressions including *red and blue*, *red or blue*, *not red*, *not blue*, *red and not blue* etc.
- For each label L_i there is a set of prototypical elements $P_i \subseteq \Omega$.
- $d : \Omega^2 \rightarrow [0, \infty)$ is a pseudo-distance satisfying $d(x, x) = 0$ and $d(x, y) = d(y, x)$ for all $x, y \in \Omega$.

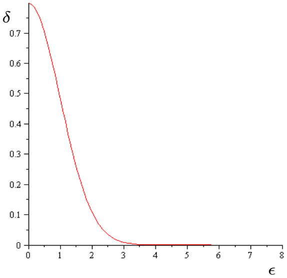
Random Set Neighbourhoods

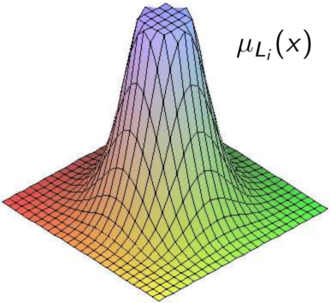
Label L_i has prototypes $P_i \subseteq \Omega$

$\mathcal{N}_{L_i}^{\epsilon} = \{x \in \Omega : d(x, P_i) \leq \epsilon\}$

Threshold ϵ is uncertain.



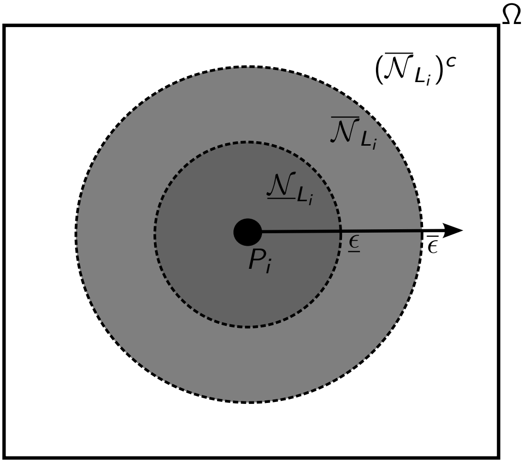




$$\mu_{L_i}(x) = P(x \in \mathcal{N}_{L_i}^{\epsilon}) = \int_{d(x,P_i)}^{\infty} \delta(\epsilon)d\epsilon$$

Lower and Upper Neighbourhoods

$$\underline{\mathcal{N}}_{L_i} = \{x \in \Omega : d(x, P_i) \leq \underline{\epsilon}\} \subseteq \overline{\mathcal{N}}_{L_i} = \{x \in \Omega : d(x, P_i) \leq \overline{\epsilon}\}$$



Imprecision Parameter

- We assume that both $\underline{\epsilon}$ and $\overline{\epsilon}$ are functions of a single parameter α taking values in $[0,1]$.
- α quantifies an agent's overall level of imprecision in their definition of labels, so that as α increases the difference between the upper extension of a label and its lower extension decreases.
- There exists an increasing function $\underline{f} : [0,1] \rightarrow [0,\infty)$ and a decreasing function $\overline{f} : [0,1] \rightarrow [0,\infty)$ such that $\underline{f} \leq \overline{f}$ and for which $\underline{\epsilon} = \underline{f}(\alpha)$ and $\overline{\epsilon} = \overline{f}(\alpha)$.

Neighbourhoods for Compound Expressions

- The following combination rules are defined to determine neighbourhoods for compound expressions:
- $\forall L_i \in LA\ \underline{\mathcal{N}}^{\alpha}_{L_i} = \{x : d(x, P_i) \leq \underline{f}(\alpha)\}$
- $\overline{\mathcal{N}}^{\alpha}_{L_i} = \{x : d(x, P_i) \leq \overline{f}(\alpha)\}$.
- $\forall \theta, \varphi \in LE\ \underline{\mathcal{N}}^{\alpha}_{\theta \wedge \varphi} = \underline{\mathcal{N}}^{\alpha}_{\theta} \cap \underline{\mathcal{N}}^{\alpha}_{\varphi}$, $\overline{\mathcal{N}}^{\alpha}_{\theta \wedge \varphi} = \overline{\mathcal{N}}^{\alpha}_{\theta} \cap \overline{\mathcal{N}}^{\alpha}_{\varphi}$.
- $\forall \theta, \varphi \in LE\ \underline{\mathcal{N}}^{\alpha}_{\theta \vee \varphi} = \underline{\mathcal{N}}^{\alpha}_{\theta} \cup \underline{\mathcal{N}}^{\alpha}_{\varphi}$, $\overline{\mathcal{N}}^{\alpha}_{\theta \vee \varphi} = \overline{\mathcal{N}}^{\alpha}_{\theta} \cup \overline{\mathcal{N}}^{\alpha}_{\varphi}$.
- $\forall \theta \in LE\ \underline{\mathcal{N}}^{\alpha}_{\neg\theta} = (\overline{\mathcal{N}}^{\alpha}_{\theta})^c$, $\overline{\mathcal{N}}^{\alpha}_{\neg\theta} = (\underline{\mathcal{N}}^{\alpha}_{\theta})^c$

Lower and Upper Membership Functions

- Let δ be a probability measure characterised by a density function on $[0, 1]$ (also denoted δ) quantifying the uncertainty about α
- **Lower Membership:**
  $\forall \theta \in LE$, $\forall x \in \Omega$, $\underline{\mu}_\theta(x) = \delta(\{\alpha : x \in \underline{\mathcal{N}}_\theta^\alpha\})$
- **Upper Membership:**
  $\forall \theta \in LE$, $\forall x \in \Omega$, $\overline{\mu}_\theta(x) = \delta(\{\alpha : x \in \overline{\mathcal{N}}_\theta^\alpha\})$
- For any expression θ, $\underline{\mathcal{N}}_\theta^\alpha$ and $\overline{\mathcal{N}}_\theta^\alpha$ are both random sets taking as values subsets of Ω.
- From this perspective $\underline{\mu}_\theta$ and $\overline{\mu}_\theta$ are the single point coverage functions of $\underline{\mathcal{N}}_\theta^\alpha$ and $\overline{\mathcal{N}}_\theta^\alpha$ respectively.

Main Results

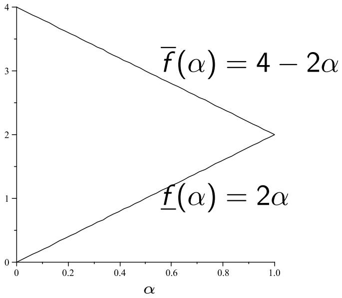
- **Theorem:** $\forall \theta \in LE$, $\forall \alpha \in [0,1]$ $\underline{\mathcal{N}}_\theta^\alpha \subseteq \overline{\mathcal{N}}_\theta^\alpha$
- and consequently $\forall \theta \in LE$, $\forall x \in \Omega$ $\underline{\mu}_\theta(x) \leq \overline{\mu}_\theta(x)$
- **Theorem:** $\forall \alpha, \alpha' \in [0,1]$ where $\alpha \leq \alpha'$ it holds that $\forall \theta \in LE$ $\underline{\mathcal{N}}_\theta^\alpha \subseteq \underline{\mathcal{N}}_\theta^{\alpha'}$ and $\overline{\mathcal{N}}_\theta^\alpha \supseteq \overline{\mathcal{N}}_\theta^{\alpha'}$.
- and consequently
- $\underline{\mu}_{\theta\wedge\varphi}(x) = \min(\underline{\mu}_\theta(x), \underline{\mu}_\varphi(x))$, $\overline{\mu}_{\theta\wedge\varphi}(x) = \min(\overline{\mu}_\theta(x), \overline{\mu}_\varphi(x))$
- $\underline{\mu}_{\theta\vee\varphi}(x) = \max(\underline{\mu}_\theta(x), \underline{\mu}_\varphi(x))$, $\overline{\mu}_{\theta\vee\varphi}(x) = \max(\overline{\mu}_\theta(x), \overline{\mu}_\varphi(x))$
- $\underline{\mu}_{\neg\theta}(x) = 1 - \overline{\mu}_\theta(x)$, $\overline{\mu}_{\neg\theta}(x) = 1 - \underline{\mu}_\theta(x)$
- **By Definition:** $\underline{\mu}_{\theta\wedge\neg\theta}(x) = 0$ and $\overline{\mu}_{\theta\vee\neg\theta}(x) = 1$ and consequently
- $\min(\underline{\mu}_\theta(x), 1 - \overline{\mu}_\theta(x)) = 0$.

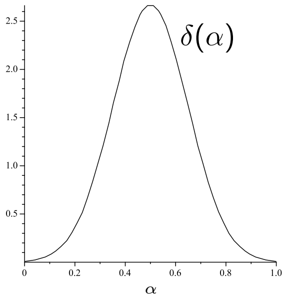
Example

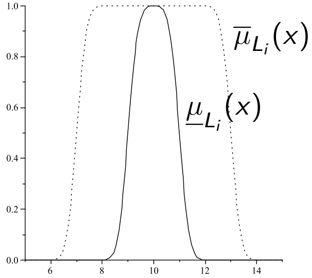
- Let $\Omega = \mathbb{R}$ and L_i be a label with prototype $P_i = \{10\}$ (i.e. L_i denotes *about 10*).
- Let $\underline{f}(\alpha) = 2\alpha$ and $\overline{f}(\alpha) = 4 - 2\alpha$ and also let δ be a gaussian distribution with mean 0.5 and standard deviation 0.15 normalised so as to have integral 1 on $[0, 1]$
- From this we have the following lower and upper neighbourhoods:
- $\underline{\mathcal{N}}^{\alpha}_{L_i} = [10 - 2\alpha, 10 + 2\alpha]$ and $\overline{\mathcal{N}}^{\alpha}_{L_i} = [6 + 2\alpha, 14 - 2\alpha]$

$$\underline{\mu}_{L_i}(x) = \begin{cases} \int_{\frac{10-x}{2}}^{1} \delta(\epsilon)d\epsilon : 8 \leq x \leq 10 \\ \int_{\frac{x-10}{2}}^{1} \delta(\epsilon)d\epsilon : 10 < x \leq 12 \\ 0 : \text{otherwise} \end{cases} \quad \overline{\mu}_{L_i}(x) = \begin{cases} \int_{0}^{\frac{x-6}{2}} \delta(\epsilon)d\epsilon : 6 \leq x \leq 10 \\ \int_{0}^{\frac{14-x}{2}} \delta(\epsilon)d\epsilon : 10 < x \leq 14 \\ 0 : \text{otherwise} \end{cases}$$

Example Continued..







Comments and Conclusions

- We have introduced a random set and prototype theory interpretation of lower and upper fuzzy membership functions.
- Random sets are defined based on lower and upper threshold distances from prototypes which are taken to be functions of a single parameter α indicating the overall level of imprecision associated with concept definition.
- Uncertainty associated with the correct level of α is modelled by a probability density function δ.
- Based on this definition lower and upper membership functions are fully truth-functional satisfying the min and max rules for conjunction and disjunction as proposed for interval fuzzy sets.