Conditional Beliefs in a Bipolar Framework

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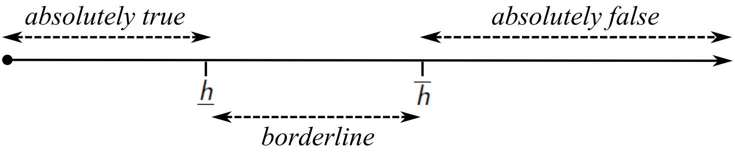
Outline of the Talk

- Kleene valuation pairs: Truth-gaps and indeterminism.
- Kleene Belief pairs: Combining uncertainty and vagueness.
- Introducing conditional belief pairs.
- Properties and relationships to other uncertainty theories.
- Conclusions.

Truth-gaps and Indeterminism

- Propositions can be *absolutely true* or *absolutely false* but there may be also be a *truth-gap*.
- Some propositions may be neither absolutely true nor absolutely false i.e. indeterminate or borderline
- For example, consider the proposition $p =$'the suspect is short'.
- In this context *short* could be defined by two height thresholds $\underline{h} \leq \overline{h}$.
- Let suspect's height be h. Then p is absolutely true if $h \leq \underline{h}$, absolutely false of $h > \overline{h}$ and borderline otherwise.



Kleene Valuation Pairs

- $\mathcal{L}$ is a language with propositional variable $\mathcal{P} = \{p_1, \ldots, p_l\}$, connectives $\neg, \wedge, \vee$ and sentences $S\mathcal{L}$.

Definition

A valuation pair is a pair of binary functions $(\underline{v}, \overline{v})$ *where* $\underline{v} : S\mathcal{L} \to \{0,1\}$, $\overline{v} : S\mathcal{L} \to \{0,1\}$ *and* $\underline{v} \leq \overline{v}$.

- For sentence $\theta \in S\mathcal{L}$, $\underline{v}(\theta) = 1$ means that θ is *absolutely true*, and $\overline{v}(\theta) = 1$ means that θ is *not absolutely false*.

Definition

A Kleene valuation pair is valuation pair $(\underline{v}, \overline{v})$ *satisfying:*
$\forall \theta, \varphi \in S\mathcal{L}$

1. $\underline{v}(\neg\theta) = 1 - \overline{v}(\theta)$, $\overline{v}(\neg\theta) = 1 - \underline{v}(\theta)$
2. $\underline{v}(\theta \wedge \varphi) = \min(\underline{v}(\theta), \underline{v}(\varphi))$ *and* $\overline{v}(\theta \wedge \varphi) = \min(\overline{v}(\theta), \overline{v}(\varphi))$
3. $\underline{v}(\theta \vee \varphi) = \max(\underline{v}(\theta), \underline{v}(\varphi))$ *and* $\overline{v}(\theta \vee \varphi) = \max(\overline{v}(\theta), \overline{v}(\varphi))$

Valuation Pairs and Three-Valued Logic

- We can also think of a valuation pair as a three-valued mapping with $\vec{v}(\theta)$ having possible values $\mathbf{t} = (1, 1)$, $\mathbf{b} = (0, 1)$ and $\mathbf{f} = (0, 0)$.
- Given the definition of Kleene valuation pairs this naturally generates the following truth-tables:

$\wedge$	**t**	**b**	**f**
t	**t**	**b**	**f**
b	**b**	**b**	**f**
f	**f**	**f**	**f**

$\vee$	**t**	**b**	**f**
t	**t**	**t**	**t**
b	**t**	**b**	**b**
f	**t**	**b**	**f**

$\neg$	
t	**f**
b	**b**
f	**t**

- Let $\mathbb{V}$ denote the set of Kleene valuation pairs on $\mathcal{L}$.

Theorem

$\forall \vec{v} \in \mathbb{V},\ \vec{v}(\varphi) = \mathbf{b}$ *if and only if* $\overline{v}(\varphi \wedge \neg\varphi) = 1$ *if and only if* $\underline{v}(\varphi \vee \neg\varphi) = 0$.

Semantic Precision: An Ordering on Precisification

- We now define *semantic precision* as a natural partial ordering on valuation pairs.

Definition

Given two valuation pairs $\vec{v}_1$ *and* $\vec{v}_2$, $\vec{v}_1 \preceq \vec{v}_2$ *if and only if* $\forall \theta \in S\mathcal{L}$, $\underline{v}_1(\theta) \leq \underline{v}_2(\theta)$ *and* $\overline{v}_1(\theta) \geq \overline{v}_2(\theta)$.

- $\vec{v}_1$ is less semantically precise than $\vec{v}_2$ if they disagree only for some set of sentences of $\mathcal{L}$, which being identified as either **t** or **f** by $\vec{v}_2$, are classified as being **b** by $\vec{v}_1$.
- One might think of $\preceq$ as ordering valuation pairs according to their relative vagueness.
- Shapiro (2006) proposed essentially the same ordering of interpretations which he refers to as *sharpening* i.e. $\vec{v}_1 \preceq \vec{v}_2$ means that $\vec{v}_2$ extends or sharpens $\vec{v}_1$.

Uncertainty and Vagueness

- Consider a combined model incorporating both indeterminism and epistemic uncertainty.
- In this context natural division of uncertainty types is:
- **Semantic Uncertainty**: This takes the form of uncertainty about what is the *correct* interpretation of $\mathcal{L}$. For example, an agent may be uncertain as to whether or not a proposition such as 'the suspect is short' is absolutely true or absolutely false even if they know suspect's height h precisely.
- This uncertainty naturally arises from the distributed manner in which language is learnt through communications with other agents across a population of interacting agents.
- **Possible Worlds Uncertainty**: Arises from a lack of knowledge concerning the state of the world. For example, being uncertain about the suspect's height h in the proposition 'the suspect is short'.

Kleene Belief Pairs

- In our current model epistemic uncertainty takes the form of uncertainty as to which is the *correct* valuation pair.
- Let w be a probability distribution defined on $\mathbb{V}$ so that $w(\vec{v})$ is the agent's subjective belief that $\vec{v}$ is the correct valuation pair for $\mathcal{L}$.

Definition

$\vec{\mu} = (\underline{\mu}, \overline{\mu})$ *is a Kleene belief pair where* $\forall\theta \in S\mathcal{L}$,
$\underline{\mu}(\theta) = w(\{\vec{v} \in \mathbb{V} : \underline{v}(\theta) = 1\})$ *and* $\overline{\mu}(\theta) = w(\{\vec{v} \in \mathbb{V} : \overline{v}(\theta) = 1\})$.

Theorem

For all $\theta, \varphi \in S\mathcal{L}$, $\underline{\mu}(\theta) \leq \overline{\mu}(\theta)$ *and the following hold:*

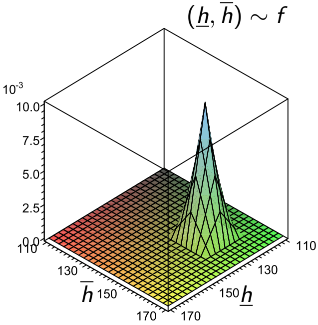
- $\underline{\mu}(\neg\theta) = 1 - \overline{\mu}(\theta)$ *and* $\overline{\mu}(\neg\theta) = 1 - \underline{\mu}(\theta)$.
- $\underline{\mu}(\theta \vee \varphi) = \underline{\mu}(\theta) + \underline{\mu}(\varphi) - \underline{\mu}(\theta \wedge \varphi)$
- $\overline{\mu}(\theta \vee \varphi) = \overline{\mu}(\theta) + \overline{\mu}(\varphi) - \overline{\mu}(\theta \wedge \varphi)$

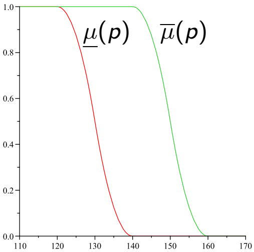
Example: The *Short* Suspect

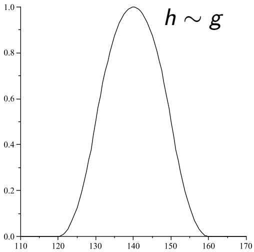
- p='the suspect is short'; if the suspects height is h then:

$$\underline{v}(p) = 1 \text{ iff } h \leq \underline{h} \text{ and}$$
$$\overline{v}(p) = 1 \text{ iff } h \leq \overline{h}$$

- If $h \sim g$ a normal distribution with mean 140 and s.d. 7 then $\underline{\mu}(p) = 0.1092$ and $\overline{\mu}(p) = 0.8908$







Conditional Belief Pairs

- We introduce conditional belief pairs based on conditional probabilities.
- New knowledge is assumed to take the form of lower and upper valuation constraints.

$$K = \{\underline{v}(\theta_1) = 1, \ldots, \underline{v}(\theta_t) = 1, \overline{v}(\varphi_1) = 1, \ldots, \overline{v}(\varphi_s) = 1\}$$

- $\mathbb{V}(K) \subseteq \mathbb{V}$ denotes the set of Kleene valuation pairs on $\mathcal{L}$ which satisfy the constraints K.

Definition

We define lower and upper conditional belief measures conditional on K as follows:

$$\underline{\mu}(\theta|K) = \frac{w(\{\vec{v} \in \mathbb{V}(K) : \underline{v}(\theta) = 1\})}{w(\mathbb{V}(K))}$$

$$\overline{\mu}(\theta|K) = \frac{w(\{\vec{v} \in \mathbb{V}(K) : \overline{v}(\theta) = 1\})}{w(\mathbb{V}(K))}$$

Results for Conditional Belief Pairs

- Consider the special cases where $K = \{\underline{v}(\varphi) = 1\}$, $K = \{\overline{v}(\varphi) = 1\}$ and $K = \{\underline{v}(\varphi) = 0, \overline{v}(\varphi) = 1\}$ for some sentence $\varphi \in S\mathcal{L}$.
- These correspond to the knowledge that $\vec{v}(\varphi) = \mathbf{t}$, $\vec{v}(\varphi) \neq \mathbf{f}$ and $\vec{v}(\varphi) = \mathbf{b}$ respectively.

Theorem

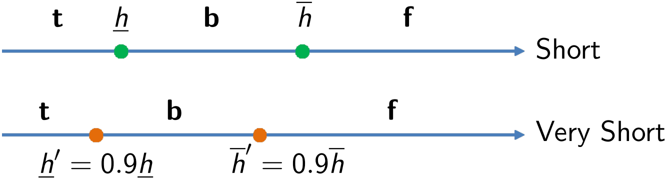
- $\underline{\mu}(\theta|\underline{v}(\varphi) = 1) = \frac{\underline{\mu}(\theta \wedge \varphi)}{\underline{\mu}(\varphi)}$ *and* $\overline{\mu}(\theta|\underline{v}(\varphi) = 1) = \frac{\overline{\mu}(\theta \vee \neg\varphi) - \overline{\mu}(\neg\varphi)}{1 - \overline{\mu}(\neg\varphi)}$
- $\underline{\mu}(\theta|\overline{v}(\varphi) = 1) = \frac{\underline{\mu}(\theta \vee \neg\varphi) - \underline{\mu}(\neg\varphi)}{1 - \underline{\mu}(\neg\varphi)}$ *and* $\overline{\mu}(\theta|\overline{v}(\varphi) = 1) = \frac{\overline{\mu}(\theta \wedge \varphi)}{\overline{\mu}(\varphi)}$
- $\underline{\mu}(\theta|\vec{v}(\varphi) = \mathbf{b}) = \frac{\underline{\mu}(\theta \vee \varphi \vee \neg\varphi) - \underline{\mu}(\varphi \vee \neg\varphi)}{1 - \underline{\mu}(\varphi \vee \neg\varphi)}$ *and*
  $\overline{\mu}(\theta|\vec{v}(\varphi) = \mathbf{b}) = \frac{\overline{\mu}(\theta \wedge \varphi \wedge \neg\varphi)}{\overline{\mu}(\varphi \wedge \neg\varphi)}$

- For $K = \{\overline{v}(\varphi) = 1\}$ the conditional measures appear to be the same as in DS-theory. But beware!

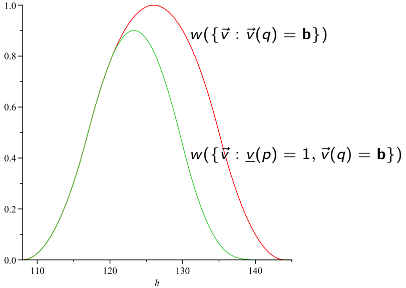
Example: Conditioning Given a Borderline Case

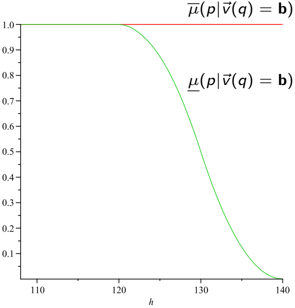
$p =$'suspect is short'

$q =$'suspect is very short'



- $\vec{v}(q) = \mathbf{b} \Rightarrow h \leq \overline{h}' < \overline{h} \Rightarrow \overline{v}(p) = 1$
- Hence, $\overline{\mu}(p|\vec{v}(q) = \mathbf{b}) = 1$.
- If $h \sim g$ then $\underline{\mu}(p|\vec{q} = b) = 0.3383$





Uncertainty about the Level of Vagueness

- Suppose an agent is only uncertain about the correct level of semantic precision for interpretation of $\mathcal{L}$.
- Formally; suppose w is non-zero only on a sequence $\vec{v}_1 \preceq \vec{v}_2 \preceq \ldots \preceq \vec{v}_k$

Theorem

In this case $\vec{\mu}$ satisfies the following properties; $\forall \theta, \varphi \in S\mathcal{L}$,

$$\underline{\mu}(\theta \wedge \varphi) = \min(\underline{\mu}(\theta), \underline{\mu}(\varphi)) \textit{ and } \overline{\mu}(\theta \wedge \varphi) = \min(\overline{\mu}(\theta), \overline{\mu}(\varphi))$$
$$\underline{\mu}(\theta \vee \varphi) = \max(\underline{\mu}(\theta), \underline{\mu}(\varphi)) \textit{ and } \overline{\mu}(\theta \vee \varphi) = \max(\overline{\mu}(\theta), \overline{\mu}(\varphi))$$

Corollary

In this case conditional belief measures obey Goguen implication:

$$\underline{\mu}(\theta|\underline{v}(\varphi) = 1) = \begin{cases} \frac{\underline{\mu}(\theta)}{\underline{\mu}(\varphi)} : \underline{\mu}(\theta) \leq \underline{\mu}(\varphi) \\ 1 : \textit{otherwise} \end{cases} \quad \overline{\mu}(\theta|\overline{v}(\varphi) = 1) = \begin{cases} \frac{\overline{\mu}(\theta)}{\overline{\mu}(\varphi)} : \overline{\mu}(\theta) \leq \overline{\mu}(\varphi) \\ 1 : \textit{otherwise} \end{cases}$$

Conclusions

- We have extended Kleene belief pairs by introducing lower and upper conditional belief pairs.
- The properties of these measures has been investigated and relationship to existing uncertainty theories highlighted.
- This extended framework is sufficiently rich to capture aspects of both stochastic and semantic uncertainty together with indeterminism in the underlying truth model.
- This can provide an enhanced model for decision making in the presence of both uncertainty and conceptual vagueness.
- Also, it can permit the definition of more flexible rules and specifications for intelligent systems.
- e.g. $\underline{\mu}(p|\vec{v}(q) = b) \geq \alpha$ for a suitable confidence level α.